Homomorphisms (D+F 1.6)

When we talk about functions between groups, it makes sense to limit our scope to functions that preserve the group operation (morphisms in the category of groups). More precisely:

Def: Let G and H be groups. A function $\varphi : G \to H$ is a <u>homomorphism</u> if $\forall \; x, y \in G$, $\underbrace{\varphi(xy)}_{\text{multiplied in } G} = \underbrace{\varphi(x)\varphi(y)}_{\text{multiplied in } H}$.

Ex: Consider the function $\varphi : \underbrace{\mathbb{Z}}_{\text{w/ addition}} \to D_{2n}$ defined

$$\varphi(a) = r^a \; (= r^{a \,(\mathrm{mod}\, n)}).$$

Then for $a, b \in \mathbb{Z}$, $\varphi(a+b) = r^{a+b} = r^a r^b = \varphi(a)\varphi(b)$, so it is a homomorphism.

Ex: Consider the function $\Psi : D_8 \to \mathbb{Z}/4\mathbb{Z}$ defined

$$\Psi(s^i r^j) = \bar{j}.$$

Is this a homomorphism?

$\Psi(sr) + \Psi(sr) = \bar{1} + \bar{1} = \bar{2}$, but

$\Psi(srsr) = \Psi(srr^3s) = \Psi(1) = \bar{0}$, so it's <u>not</u> a homomorphism.

Ex: Define the map $\exp : \mathbb{R} \to \mathbb{R}^+$ by $\exp(x) = e^x$.

$\mathbb{R}$ w/ addition; $\mathbb{R}^+$ w/ multiplication (check that this is a group!)

Then $\exp(x+y) = e^{x+y} = e^x e^y = \exp(x)\exp(y)$, so it's a homomorphism.

In fact, it's a bijection as well! We have an inverse homomorphism: $\ln : \mathbb{R}^+ \to \mathbb{R}$, the natural logarithm.

Notice that this means $\mathbb{R}^+$ and $\mathbb{R}$ have the same set of elements (renamed), and the same exact group structure. That is, they are "isomorphic":

Def: An <u>isomorphism</u> of groups is a bijective homomorphism.

If $\varphi : G \to H$ is an isomorphism, we say G and H are <u>isomorphic</u>, denoted $G \cong H$.

Note that if G is any group, the identity $\mathrm{id} : G \to G$ is an isomorphism, but not necessarily the only isomorphism $G \to G$:

Ex: Define $\varphi : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ by

$$\begin{aligned} 0 &\mapsto 0 \\ 1 &\mapsto 3 \\ 2 &\mapsto 2 \\ 3 &\mapsto 1 \end{aligned}$$

This is a nonidentity isomorphism.

In fact:

Claim: G an abelian group, $\varphi: G \to G$ defined $\varphi(x) = x^{-1}$ is an isomorphism. (In fact, if and only if!)

Pf: $\varphi(xy) = (xy)^{-1} \underset{\substack{\uparrow \\ \text{see HW}}}{=} x^{-1}y^{-1} = \varphi(x)\varphi(y)$. $\square$

Note: If $G \cong H$, then any property of G that depends only on the group structure will also hold for H. e.g.

- $|G| = |H|$
- G is abelian $\Leftrightarrow$ H abelian
- $x \in G$, $|x| = |\varphi(x)|$
- $G' \leq G \Leftrightarrow \varphi(G') \leq H$.

Ex: The quaternion group, Q_8 is defined

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

where
- $1 \cdot a = a \cdot 1 = a \quad \forall \ a \in Q_8$
- $(-1)(-1) = 1$, $(-1) \cdot a = a \cdot (-1) = -a \quad \forall \ a \in Q_8$.
- $i^2 = j^2 = k^2 = -1$
- $i \cdot j = k$, $j \cdot k = i$, $k \cdot i = j$
- $j \cdot i = -k$, $k \cdot j = -i$, $i \cdot k = -j$

Note that $|\pm i| = |\pm j| = |\pm k| = 4$. In particular, no element has order 8, so $Q_8 \not\cong \mathbb{Z}/8\mathbb{Z}$.